LAWS OF LARGE NUMBERS

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LAWS OF LARGE NUMBERS

- In probability theory, the law of large numbers (LLN) is a theorem that describes the result of performing the same experiment a large number of times
- According to the law
  - the average of the results obtained from a large number of trials should be close to the expected value and tends to become closer to the expected value as more trials are performed
- The LLN is important because it guarantees that even random events with a large number of trials may return stable long-term results

LAWS OF LARGE NUMBERS

Three important LLNs are discussed:

- **Tchebycheff's Inequality**
- **Bernoulli's Law of Large Numbers**
- **Weak Law of Large Numbers (WLLN)**

TCHEBYCHEFF'S INEQUALITY

(CHEBYCHEV'S INEQUALITY)

Let X be a R.V. with mean μ and S.D. σ, then for any positive number 't',

$$P[|X-\mu| \geq t\sigma] \leq \frac{1}{t^2}$$

OR

$$P[|X-\mu| < t\sigma] \geq 1 - \frac{1}{t^2}$$

TCHEBYCHEFF'S INEQUALITY

Proof: Let X be a continuous R.V.

- with p.d.f. f(x), mean μ and S.D. σ
- then, Var (X)= $E[X-E(X)]^2$
- That is,
- σ^2 = $E[X-\mu]^2$ = $\int_{x:-\infty}^{\infty} (x-\mu)^2 f(x)dx$

$$= \int_{x:-\infty}^{\mu-t\sigma} (x-\mu)^2 f(x)dx + \int_{x:\mu-t\sigma}^{\mu+t\sigma} (x-\mu)^2 f(x)dx + \int_{x:\mu+t\sigma}^{\infty} (x-\mu)^2 f(x)dx$$

$$\geq \int_{x:-\infty}^{\mu-t\sigma} (x-\mu)^2 f(x)dx + \int_{x:\mu+t\sigma}^{\infty} (x-\mu)^2 f(x)dx$$

(as $\int_{x:\mu-t\sigma}^{\mu+t\sigma} (x-\mu)^2 f(x)dx$ always, since the integrand is non-negative)

TCHEBYCHEFF'S INEQUALITY

- $\sigma^2 \geq \int_{x:-\infty}^{\mu-t\sigma} (x-\mu)^2 f(x)dx + \int_{x:\mu+t\sigma}^{\infty} (x-\mu)^2 f(x)dx$
- In the first integral, $x \leq \mu - t\sigma$, that is, $\mu - x \geq t\sigma$, therefore, $(x-\mu)^2 \geq t^2\sigma^2$.
- In the second integral, $x \geq \mu + t\sigma$, yielding $(x-\mu)^2 \geq t^2\sigma^2$.
- That is, in both the integrals, the least value of $(x-\mu)^2$ is $t^2\sigma^2$.
- Hence replacing $(x-\mu)^2$ by its least value $t^2\sigma^2$ does not alter the existing inequality.
- That is, $\sigma^2 \geq \int_{x:-\infty}^{\mu-t\sigma} t^2\sigma^2 f(x)dx + \int_{x:\mu+t\sigma}^{\infty} t^2\sigma^2 f(x)dx$

TCHEBYCHEFF'S INEQUALITY

- That is, $\sigma^2 \geq \int_{x:-\infty}^{\mu-t\sigma} t^2\sigma^2 f(x)dx + \int_{x:\mu+t\sigma}^{\infty} t^2\sigma^2 f(x)dx$
- $= t^2\sigma^2\{ \int_{x:-\infty}^{\mu-t\sigma} f(x)dx + \int_{x:\mu+t\sigma}^{\infty} f(x)dx$
- $= t^2\sigma^2 \{ P[-\infty < X \leq \mu-t\sigma]+P[\mu+t\sigma \leq X < \infty]\}$
- $= t^2\sigma^2 \{ P[-\infty < X-\mu \leq -t\sigma]+P[t\sigma \leq X-\mu < \infty]\}$
- $= t^2\sigma^2 P[| X -\mu| \geq t\sigma]$
- That is, $\sigma^2 \geq t^2\sigma^2 P[| X -\mu| \geq t\sigma]$,
- implies, $P[|X-\mu| \geq t\sigma] \leq \frac{1}{t^2}$; the first form!

TCHEBYCHEFF'S INEQUALITY

- $P[|X-\mu| \geq t\sigma] \leq \frac{1}{t^2}$

- Multiplying both sides with -1;

  $$-P[|X-\mu| \geq t\sigma] \geq -\frac{1}{t^2}$$

- Adding '1' to both sides; $1-P[|X-\mu| \geq t\sigma] \geq 1-\frac{1}{t^2}$
- That is, $P[|X-\mu| < t\sigma] \geq 1-\frac{1}{t^2}$

- Hence, the second form of theorem holds.
- Note: If the R.V. is discrete type, the theorem can be proved by applying summation instead of integration.

TCHEBYCHEFF'S INEQUALITY

Merits of the Inequality

- It gives an upper bound (lower bound) to the probability of a R.V. deviating from its mean more than (less than) 't' times the S.D. σ.
- It is valid for both discrete and continuous variables.
- It is applicable to all R.V.s for which mean and S.D. known, without knowing its probability distribution.
- It justifies the importance of S.D. as a measure of dispersion.
- If the moments only up to second order exist, this inequality provides the possible best result.

TCHEBYCHEFF'S INEQUALITY

Demerits of the Inequality

- It will give a useful result only if the probability distribution of the variable is unknown.
- If $t < 1$, the upper bound (lower bound) obtained will be more than 1 (less than 0) and hence is of no use.
- The upper bound (lower bound) obtained in most cases is much larger (smaller) than the true probability.

BERNOULLI'S LAW OF LARGE NUMBERS

- Consider a random experiment with only two possible outcomes success and failure.
- Let 'p' be the probability of success which remains unchanged from trial to trial.
- Consider n independent trials of the experiment.
- Let X be the number of success in these n trials. Then for any ε > 0,
- $P[\ |\frac{X}{n} - p| \geq \varepsilon] \to 0$ as $n \to \infty$

OR

- $P[\ |\frac{X}{n} - p| < \varepsilon] \to 1$ as $n \to$

BERNOULLI'S LAW OF LARGE NUMBERS

Proof: Here X ~ b (n, p)

E(X) = np and V(X) = npq; q= 1-p

Consider $\frac{X}{n}$; Here μ = E($\frac{X}{n}$) = p

$\sigma^2 = V\left(\frac{X}{n}\right) = \frac{npq}{n^2} = \frac{pq}{n}$

By Tchebycheff's Inequality, P[|X-μ| ≥ tσ] ≤ $\frac{1}{t^2}$

On substitution, $P\left[\left|\frac{X}{n} - p\right| \geq t\sqrt{\frac{pq}{n}}\right] \leq \frac{1}{t^2}$

Let $\varepsilon = t\sqrt{\frac{pq}{n}}$, then $\frac{1}{t^2} = \frac{pq}{n\varepsilon^2}$;

Hence, **$P\left[\left|\frac{X}{n} - p\right| \geq \varepsilon\right] \leq \frac{pq}{n\varepsilon^2}$**

BERNOULLI'S LAW OF LARGE NUMBERS

Hence, **P[$|\frac{X}{n}$ - p| ≥ ε] ≤** $\frac{pq}{n\varepsilon^2}$

- 'pq' is maximum when p = q, with maximum value 1/4 . That is, pq ≤1/4 . Hence on substituting the maximum value of 'pq' in the above inequality, the existing inequality does not alter.
- Therefore, **P[$|\frac{X}{n}$- p| ≥ ε] ≤** $\frac{1}{4n\varepsilon^2}$
- As n is very large $\frac{1}{4n\varepsilon^2}$ tends to 0. Hence, the theorem
- *The alternate form is in terms of the complement probability*

BERNOULLI'S LAW OF LARGE NUMBERS

- The law states that, as the number of repetitions (n) increases indefinitely, the relative frequency of an event *converges in probability* to the probability (p) of the event.
- This phenomenon is called *statistical regularity*, which is the background of the *frequency definition of probability*.

WEAK LAW OF LARGE NUMBERS

- Let $X_1, X_2, \ldots, X_n$ be n random variables having $\mu_i = E(X_i)$ for i = 1, 2, …, n.
- Let $S_n = \sum_{i=1}^{n} X_i$ $M_n = E(S_n)$, and $B_n = Var(S_n)$
- Then for any ε > 0,

$P[\ |\frac{S_n}{n} - \frac{M_n}{n}| \geq \varepsilon] \rightarrow$ as $n \rightarrow \infty$, provided $\frac{B_n}{n^2} \rightarrow 0$

WEAK LAW OF LARGE NUMBERS

Proof: Here $S_n = \sum_{i=1}^{n} X_i$ Consider $\frac{S_n}{n}$;

- Here $\mu = \frac{M_n}{n}$ and $\sigma^2 = \frac{B_n}{n^2}$
- By Tchebycheff's Inequality, $P[|X-\mu| \geq t\sigma] \leq \frac{1}{t^2}$
- On substitution, $P\left[\left| \frac{S_n}{n} - \frac{M_n}{n} \right| \geq t\sqrt{\frac{B_n}{n^2}} \right] \leq \frac{1}{t^2}$
- Let $\varepsilon = t\sqrt{\frac{B_n}{n^2}}$, then $\frac{1}{t^2} = \frac{B_n}{n^2\varepsilon^2}$;
- As n is large, the theorem holds.
- When X_i follows Bernoulli distribution for i= 1, 2, …, n, then WLLN reduces to Bernoulli's law.

Thank you